Quantized vortices and collective oscillations of a trapped Bose condensed gas

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(February 1, 2008)

Using a sum rule approach we calculate the frequency shifts of the quadrupole oscillations of a harmonically trapped Bose gas due to the presence of a quantized vortex. Analytic results are obtained for positive scattering lengths and large N where the shift relative to excitations of opposite angular momentum is found to be proportional to the quantum circulation of the vortex and to decrease as $N^{-2/5}$. Results are also given for smaller values of N covering the transition between the ideal gas and the Thomas-Fermi limit. For negative scattering lengths we predict a macroscopic instability of the vortex. The splitting of the collective frequencies in toroidal configurations is also discussed.

PACS numbers: 03.75.Fi, 05.30.Jp, 32.80.Pj, 67.40.Db

After the experimental realization of Bose-Einstein condensation in dilute atomic gases, the study of the collecive excitations of these unique inhomogeneus quantum systems has been the object of both experimental [1–4] and theoretical [5,6] work (for an update list of references see, for example, [7]). These oscillations are characterized by proper quantum numbers, reflecting the symmetry of the confining potential. In an axially symmetric trap the third component of angular momentum is a natural quantum number and if the system is in a time reversal invariant configuration, elementary excitations carrying opposite angular momentum are degenerate. This degeneracy is in general removed if time reversal symmetry is broken. The purpose of this work is to describe the frequency shifts produced by the presence of a quantized vortex. In view of the important role played by vortices in understanding the mechanisms of superfluidity, the possibility of their spectroscopic diagnostics is highly interesting since the measurements of collective frequencies can be carried out with high precision in these systems.

The occurrence of splitting in the presence of a vortex can be simply understood by noting that the average velocity flow associated with the collective oscillation can be either parallel or opposite to the vortex flow, depending on the sign of the angular momentum carried by the excitation. This produces a shift of the collective frequency of order $\delta\omega/\omega \sim v/c$ where $v \sim 1/R$ is the velocity of the vortex flow and c is the sound velocity. In a trapped Bose gas c increases linearly with the radius R of the condensate, while ω is practically independent of R, so one expects relative shifts of the order of $1/R^2$. These effects are larger than the typical corrections to the Thomas-Fermi limit due to finite size effects which, in the absence of the vortex, behave like $logR/R^4$ [8]. The problem of the frequency shift produced by a quantized vortex has been already the object of theoretical work using semiclassical approaches based on a large N expansion [9], as well as by full numerical solution of the linearized equations of motion [10].

In this letter we develop a sum rule approach [11] which is expected to provide exact results for the splitting of the excitation frequencies in large systems. The method can be also applied to calculate the frequency shifts for small values of N as well as for negative scattering lengths. Let us introduce the strength distribution function

$$S_\pm(E) = \sum_n |\langle n\,|F_\pm|\,0\rangle|^2 \delta\big(E - \hbar\omega_{n0}\big) \tag{1}$$

relative to the operators $F_\pm = \sum_{k=1}^N f_\pm(\mathbf{r}_k)$ carrying opposite angular momenta. In equation (1) $\hbar\omega_{n0} = (E_n - E_0)$ are the excitation energies relative to the eigenstates $|n\rangle$ of the Hamiltonian

$$H = \sum_i \left(\frac{1}{2M}p_i^2 + V_{\rm ext}(\mathbf{r}_i)\right) + g\sum_{i<j}\delta(\mathbf{r}_i - \mathbf{r}_j)\ , \tag{2}$$

which describes N interacting bosons confined by an external potential. The potential $V_{\rm ext}(\mathbf{r}) = M(\omega_\perp^2 r_\perp^2 + \omega_z^2 z^2)/2$, with $r_\perp^2 = x^2 + y^2$, is assumed to be axially symmetric, and the interatomic force is a contact 2-body interaction whose coupling constant $g = 4\pi\hbar^2 a/M$ is fixed by the s-wave scattering length a.

In the following we will focus on the collective oscillations of low multipolarity which are easily excited in experiments by suitable modulation of the harmonic trap. For the quadrupole case we will consider the modes excited by the operators

$$f_\pm = (x \pm iy)^2 \tag{3}$$

and

$$f_\pm = (x \pm iy)z\ , \tag{4}$$

carrying angular momentum $m = \pm 2$ and $m = \pm 1$ respectively (here and in the following we will identify m with the third component of angular momentum of the elementary excitation). Only excitations with $m \neq 0$ are relevant for the present discussion.

In the absence of vortices the ground state has zero angular momentum and, for large N and positive scattering lengths, the collective states excited by the operators (3) and (4) are well described by hydrodynamic theory of superfluids. This yields [6] the result $\omega_\pm = \sqrt{2}\omega_\perp$ and $\omega_\pm = \sqrt{\omega_\perp^2 + \omega_z^2}$ for the $m=\pm 2$ and $m=\pm 1$ frequencies. Notice that these results differ from the ideal gas predictions, $\omega_\pm = 2\omega_\perp$ and $\omega_\pm = \omega_\perp + \omega_z$, as a consequence of interaction effects which suppress the contribution of the kinetic energy pressure term in the equations of motion. Differently from the quadrupole excitations, the frequencies of the dipole modes excited by $f_\pm = x \pm iy$ are instead unaffected by two body interactions and are given by $\omega_\pm = \omega_\perp$. This behavior is the consequence of the translational invariance of the interatomic force which cannot affect the motion of the center of mass, even in the presence of a vortex.

The moments

$$m_p^\pm = \int_0^\infty dE\big(S_+(E) \pm S_-(E)\big)E^p \tag{5}$$

of the strength distribution (1) can be calculated using closure relations. For the lowest moments we find the results:

$$m_0^- = \langle [F_-, F_+] \rangle = 0 \tag{6}$$

$$m_1^+ = \langle \big[F_-, [H, F_+]\big] \rangle = \frac{N\hbar^2}{M} \langle\, |\boldsymbol{\nabla} f_+|^2 \rangle \tag{7}$$

$$m_2^- = \langle \big[[F_-, H], [H, F_+]\big] \rangle = N \langle [j_-, j_+] \rangle\ , \tag{8}$$

where the average $\langle\ \rangle$ is taken on the state $|0\rangle$ which may or may not contain a vortex and we have used the property $F_+^\dagger = F_-$.

The first commutator (6) vanishes because the operators F_+ and F_-, depend only on the spatial coordinates. The double commutator (7) is the analog of the f-sum rule [12] and gets contribution only from the kinetic energy term since both the external potential and the two-body interaction commute with $F_\pm$. Finally the current operators $j_\pm = [p^2/2M, f_\pm]$ entering the third sum rule are defined by

$$j_\pm = \frac{\hbar}{2Mi} \boldsymbol{\nabla} f_\pm(\mathbf{r}) \cdot \mathbf{p} + h.c.\ , \tag{9}$$

where $\mathbf{p}$ is the usual momentum operator. Evaluation of the sum rules m_1^+ and m_2^- is straightforward in the case of the quadrupole operators (3) and (4). For $m=\pm 2$ we find the result

$$m_1^+ = \frac{8\hbar^2}{M} N \langle r_\perp^2 \rangle \tag{10}$$

$$m_2^- = \frac{16\hbar^3}{M^2} N \langle l_z \rangle = \frac{16\hbar^4}{M^2} N\kappa\ , \tag{11}$$

where l_z is the zth component of the angular momentum operator and κ is the quantum of circulation of the vortex ($\kappa = \pm 1, \pm 2, ...$). Of course in the absence of vortices the sum rule m_2^- vanishes. Notice that results (10-11) are formally independent of the choice of the external potential as well as of the two-body interaction.

The results for the moments m_0^-, m_1^+ and m_2^- can be used to calculate the shift of the collective frequencies when the number of atoms in the trap is large and a is positive. In fact in this limit, where the behavior of the system is properly described by hydrodynamic theory of superfluids, one expects that the moments calculated above will be exhausted by two modes with frequency $\omega_\pm$ excited, respectively, by the operators $F_\pm$:

$$S_\pm(E) = \sigma^\pm \delta(E - \hbar\omega_\pm)\ , \tag{12}$$

where $\sigma^\pm$ are the corresponding strengths. Assumption (12) is equivalent to a Bijl-Feynman ansatz [13] often used to describe the collective excitations in interacting many-body systems. In the case of superfluid helium it provides an exact description of the excitation spectrum in the phonon regime (for a recent discussion on sum rules and collective excitations in Bose superfluids see, for example, [14]).

Let us discuss the consequence of the vanishing of the m_0^- moment (6). With assumption (12) for the strength distribution one immediately finds the result $\sigma^+ = \sigma^-$ and the splitting between the two frequencies can be directly written as

$$\hbar(\omega_+ - \omega_-) = m_2^- / m_1^+\ . \tag{13}$$

Use of (10-11) then yields the relevant result

$$\omega_+ - \omega_- = \frac{2}{M} \frac{\langle l_z \rangle}{\langle r_\perp^2 \rangle} = \frac{7\omega_\perp \kappa}{\lambda^{2/5}} \left(15 \frac{Na}{a_\perp}\right)^{-2/5} \tag{14}$$

for the $m = \pm 2$ modes, where $a_\perp = \sqrt{\hbar/M\omega_\perp}$ is the oscillator length in the radial direction, while $\lambda = \omega_z/\omega_\perp$ characterizes the deformation of the harmonic trap. The same calculation can be carried out for the modes excited by the $m=\pm 1$ quadrupole operators (4). In this case the result is

$$\omega_+ - \omega_- = \frac{2}{M} \frac{\langle l_z \rangle}{\langle r_\perp^2 + 2z^2 \rangle} = \frac{7\omega_\perp \kappa \lambda^{8/5}}{1 + \lambda^2} \left(15 \frac{Na}{a_\perp}\right)^{-2/5}. \tag{15}$$

In the last equalities of (14-15) we have used the Thomas-Fermi approximation [15] to evaluate the square radii of the condensate. Notice that for spherical trapping the splitting between the $m = \pm 2$ frequencies is twice the splitting between the $m = \pm 1$ modes. For large N the shifts become smaller and smaller showing that that in this limit the effects associated with the current of the vortex are small corrections to the collective flow of the oscillation. Neverthless the splittings can be sizable. For example using a spherical configuration with $a/a_\perp = 10^{-3}$ and $N = 10^6$ and taking one quantum of circulation ($\kappa = 1$) one finds that the relative shift $(\omega_+ - \omega_-)/\omega$ of the

$m = \pm 2$ states is about 10%, having used the large N result $\sqrt{2}\omega_\perp$ for the average frequency ω. This shift is much larger than the typical experimental uncertainties in the measurements of the collective frequencies [1–4].

It is worth pointing out that the frequency shift due to the vortex is exhibited by the quadrupole excitations, but not by the dipole modes excited by the operators $f_\pm = x \pm iy$. In fact in the dipole case the operators $j_\pm = \hbar(p_x \pm iy)/Mi$ commute and the sum rule m_2^- identically vanishes. As a consequence one finds $\omega_+ = \omega_-$ as expected from general arguments. Notice however that in addition to the above modes excited by the center of mass operators, another dipole mode, localized near the core of the vortex, has been predicted [10] to occur with frequency different from $\omega_\perp$. It has been recently suggested [16] that this mode could play an important role in driving the instability of the vortex.

The above results for the shift of the quadrupole frequencies hold for large N, where the assumption that the operators f_+ and f_- excite a single mode is justified. When the adimensional parameter $Na/a_\perp$ becomes small, this assumption is no longer valid. In particular, in the limit of a non interacting gas, a vortex with quantum circulation $\kappa = +1$ corresponds to putting all the atoms in the 1 p state ($l_z = +1$) of the harmonic oscillator hamiltonian and the $m = -2$ operator f_- can give rise to $\delta\omega = 2\omega_0$ as well as to $\delta\omega = 0\omega_0$ transitions. Viceversa the operator f_+ gives rise only to $\delta\omega = 2\omega_0$ transitions. The corresponding strengths are, respectively, $\sigma^-_{\rm up} = 2a_\perp^4 N$, $\sigma^-_{\rm down} = 4a_\perp^4 N$ and $\sigma^+ = 6a_\perp^4 N$.

In order to study the transition from the noninteracting to the large N regime we have to remove the single-mode assumption (12) and hence we need the knowledge of additional moments of the distribution function. The moments m_3^+, m_4^- and m_5^+ can be easily calculated for the quadrupole operators. For example, for $m = \pm 2$, we find

$$m_3^+ = \frac{16\hbar^4\omega_\perp^2}{M} N\langle r_\perp^2\rangle \left[1 + \frac{E_{{\rm kin}_\perp}}{E_{{\rm ho}_\perp}}\right] \tag{16}$$

$$m_4^- = \frac{64\hbar^5\omega_\perp^2}{M^2} N\langle l_z\rangle = \frac{64\hbar^6\omega_\perp^2}{M^2} N\kappa \tag{17}$$

$$m_5^+ = \frac{32\hbar^6\omega_\perp^4}{M} N\langle r_\perp^2\rangle \left[1 + 3\frac{E_{{\rm kin}_\perp}}{E_{{\rm ho}_\perp}} + \frac{\tilde{V}_{\rm int}}{16E_{{\rm ho}_\perp}}\right], \tag{18}$$

where $E_{{\rm kin}_\perp}$ and $E_{{\rm ho}_\perp}$ are the radial contributions to the kinetic and oscillator energies respectively, and

$$\tilde{V}_{\rm int} = g\, a_\perp^4 \int d\mathbf{r} \left[\frac{8\kappa^2 n_0^2}{r_\perp^4} + \nabla_\perp^2 n_0 \left(\frac{|\boldsymbol{\nabla}_\perp n_0|^2}{n_0} - \nabla_\perp^2 n_0\right)\right]$$

is the contribution to m_5^+ arising from two-body interactions where n_0 is the density of the condensate and κ is the quantum of circulation of the vortex. Result (16) for m_3^+ permits to obtain directly the asymptotic behavior of the quadrupole frequency. In fact in the large N limit the kinetic energy term in (16) is negligible and the ratio $(m_3^+/m_1^+)^{1/2}$ approaches the hydrodynamic result $\sqrt{2}\omega_\perp$. Result (17) for m_4^- provides a crucial check of the validity of (14). In fact one can see that in the large N limit the single-mode approximation (12), with the dispersion law

$$\omega_\pm = \sqrt{2}\omega_\perp \pm \Delta\omega\ , \tag{19}$$

where $\Delta\omega = \omega_+ - \omega_-$ is given by (14), is consistent with result (17) for the sum rule m_4^+, up to effects linear in $\Delta\omega$. Differently from the other sum rules, m_5^+ depends explicitly on the two body interaction. This contribution is very sensitive to the core region of the vortex and is important to describe the crossover from the noninteracting to the Thomas-Fermi limit.

We have calculated numerically the sum rules $m_p^\pm$ with $p = 1, .., 5$ using the solution of the Gross-Pitaevskii equation for a vortex of quantum circulation $\kappa = +1$ [17]. These sum rules are then used to evaluate the quadrupole energies and strengths, by assuming that the operator f_- excites two states with different energy, in analogy with the behavior of the noninteracting model. The results for a spherical potential are reported in Figure 1. where, for semplicity, we have plotted only the low frequency mode excited by f_-. The strength relative to the high frequency mode excited by f_- in fact vanishes rapidly when N increases and hence this mode is not physically relevant, except for small values of N. In the Figure we also report the average energy $(m_3^+/m_1^+)^{1/2}$ calculated in the absence of vortices. This energy turns out to be very close to the exact numerical solution of the linearized Gross-Pitaevskii equation (differences are always smaller than 0.5%). Notice that the frequency $\omega^-_{\rm down}$ corresponds to a rigorous upper bound to the lowest frequency of modes excited by $f_\pm$.

The frequencies $\omega_\pm$ start respectively from $2\omega_\perp$ and $0\omega_\perp$ and approach the value $\sqrt{2}\omega_\perp$ for large N. In the same limit the strengths $\sigma^\pm$ tend to the asymptotic value $\sigma^+ = \sigma^- = 4\sqrt{2}\ a_\perp^4 N\lambda^{2/5}(15Na/a_\perp)^{2/5}/7$. The Figure shows that for large $Na/a_\perp$ the asymptotic dispersion relation (19) well reproduces the behavior of the two collective frequencies. Notice that for such values of $Na/a_\perp$ the strengths $\sigma^\pm$ practically coincide, confirming the validity of the single mode assumption (12). The small asymmetry of the calculated frequencies with respect to the hydrodynamic value is a consequence of the fact that for the values of $Na/a_\perp$ reported in the figure the excitation energy differs from $\sqrt{2}\omega_\perp$ even in the absence of vortices.

We have also explored the case of negative scattering lengths. In this case we find that the excitation energy of the lowest mode becames negative revealing that the vortex is unstable against macroscopic fluctuations of the density. The dispersion law for small value of the parameter a is found to be

$$\omega^-_{\rm down} = \omega_\perp \left(\frac{Na}{a_\perp}\right)\sqrt{\frac{\lambda}{2\pi}} \tag{20}$$

and changes sign with a.

It is finally interesting to discuss how the above results are modified by changing the geometry of the problem. This is important because toroidal configurations are expected to suppress the mechanisms of instability of the vortex [16]. A useful way to pin a vortex might be achieved with a thin laser beam stabilizing the core of the vortex along the z-axis. The thickness d of the beam can be a few microns. This is larger than the size of the core, fixed by the coherence length $\xi = a_\perp^2/R$, but can be significantly smaller than the size of the condensate R. In this case the structure of the core of the vortex will be significantly modified by the pinning, but the macroscopic behaviour of the collective excitations and in particular results (14-15) for the splitting, will be modified only in a minor way. An estimate of this effect can be obtained by calculating the change of $\langle r_\perp^2 \rangle$ due to the presence of the repulsive potential generated by the laser beam. We expect small effects if $d \ll R$.

A quite different behaviour is achieved by choosing a ring geometry. In the case of an ideal ring of radius R the problem is analytically soluble using Bogoliubov theory [16]. In this case the natural excitation operators have the form $f_\pm = exp(\pm im\phi)$ where ϕ is the azimuthal angle and $\pm m$ is the angular momentum carried by the excitation. The kinetic energy operator can be replaced by $(-\hbar^2/2MR^2)\partial^2/\partial\phi^2$ and the sum rules (7) and (8) become $m_1^+ = \hbar^2 m^2/MR^2$ and $m_2^- = 2\hbar^3 m^3 \langle l_z \rangle / M^2 R^4$. Using the Bijl-Feynman ansatz (12) one immediately finds $\omega_+ - \omega_- = 2\hbar m\kappa/MR^2$, where κ is the quantum of circulation of the vortex, in agreement with the results recently discussed in [16]. Notice that for large R the lowest collective excitations in the ring geometry correspond to one-dimensional compression waves with dispersion

$$\omega = \frac{c\,|m|}{R} \pm \frac{\hbar\,\kappa\,m}{MR^2} \tag{21}$$

where c is the sound velocity. These frequencies, which are the analog of (19), coincide with the ones calculated for a system at rest in a frame rotating with angular velocity $\omega = \hbar\kappa/MR^2$.

After completing this paper we received a preprint by A.A. Svidzinsky and A.L. Fetter (cond-mat/9803181), based on a hydrodynamic description of normal modes. Similarly to the results of [9], this work predicts a frequency shift also for the dipole mode. The origin of this discrepancy remains to be understood.

Useful discussions with F. Dalfovo, A.L. Fetter, M. Gunn and N. Wilkin-Gunn are acknowledged. This work was supported by the BEC advanced research project of INFM.

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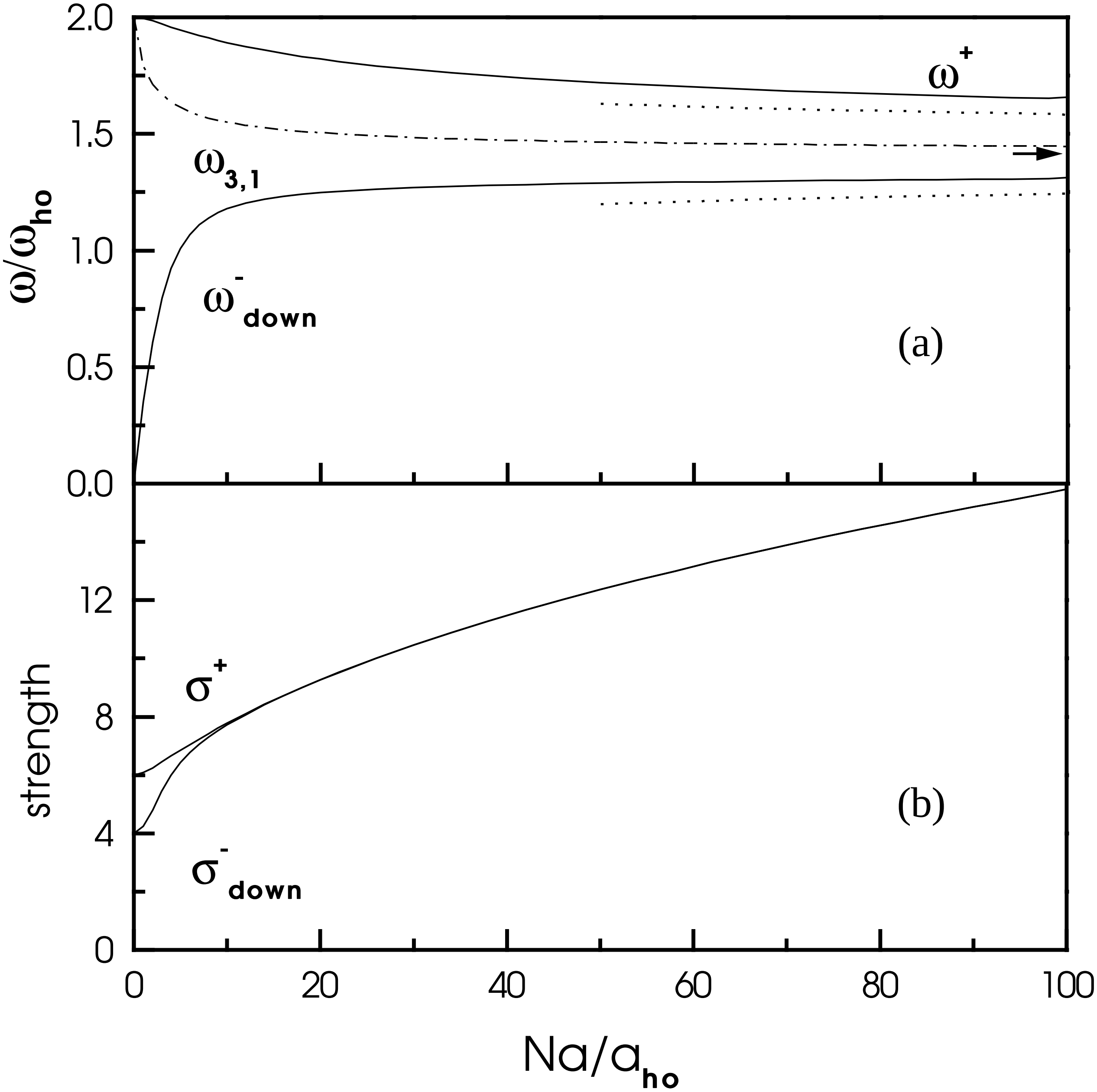
FIGURE CAPTION:

Frequencies (a) and strengths (b) relative to the $m = \pm 2$ quadrupole modes in the presence of a $\kappa = 1$ vortex, as a function of $Na/a_\perp$ for a spherical trap ($\omega_{\rm ho} = \omega_\perp$, $a_{\rm ho} = a_\perp$, $\lambda = 1$). The dotted lines correspond to the large N behavior (19). The arrow indicates the Thomas-Fermi limit $\omega = \sqrt{2}\omega_\perp$. Dashed-dots correspond to the ratio $(m_3^+/m_1^+)^{1/2}$ without vortex. Strengths are given in units of $a_\perp^4 N$.



2.0
1.5
1.0
0.5
0.0
ω/ω_ho
ω^+
ω_3,1
ω^-_down
(a)
12
8
4
0
strength
σ^+
σ^-_down
(b)
0
20
40
60
80
100
Na/a_ho